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SOME BOUNDS FOR THE NUMBER OF COINCIDENCES OF MORPHISMS BETWEEN CLOSED RIEMANN SURFACES

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ABSTRACT

We give some bounds for the number of coincidences of two morphisms between given compact Riemann surfaces (complete complex algebraic curves) which generalize well known facts about the number of fixed points of automorphisms. In the particular case in which both surfaces are hyperelliptic, our results permit us to obtain a bound for the number of morphisms between them. The proof relies on the idea, first used by Schwarz in the case of automorphisms, of representing a morphism by its action on the set of Weierstrass points.

Let M be a compact Riemann surface (complete complex algebraic curve) of genus $g \geq 2$, and $\tau\colon M \to M$ an automorphism different from the identity. Then it is well known (see e.g. [F–K]) that τ has at most $2g+2$ fixed points and that this bound is attained if and only if M is hyperelliptic and τ is the hyperelliptic involution.

For two different morphisms $f_i\colon M \to M'$ $(i = 1,2)$ of degrees d_i between compact Riemann surfaces of genera g and g' respectively, one can show that the number of **coincidences**, that is, the number of points at which f_1 and f_2 agree, is bounded by $d_1 + d_2 + 2g'\sqrt{d_1 d_2}$. In [F–G] we obtain such a bound for the number of coincidences suitably counted, and give a precise characterization of the case when this bound is attained, for $g' \geq 2$, which agrees with the classical one when the morphisms are, in fact, isomorphisms. In proving these facts we work within the framework of Lefschetz's theory of fixed points.

On the other hand, by composing with meromorphic functions of M', it is possible to obtain bounds which only depend on the degrees of the meromorphic

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function and the morphisms. In this paper we develop this idea: using the meromorphic functions associated to the Weierstrass points, in Theorem 1.5 we recover the characterization for the largest number of coincidences, that is, we prove that the maximum number of coincidences occurs when M' is hyperelliptic and $f_1 = J' \circ f_2$, with J' the hyperelliptic involution. We also observe that, just as in the case of automorphisms, there is a gap between this number and the next possible value, $g'(d_1 + d_2)$.

This method has the advantage of allowing us to obtain better results in some special cases. For example, when M' has an automorphism τ, composing the morphisms f_i with the natural projection to the quotient surface $X = M'/\langle\tau\rangle$ of genus γ, we obtain in Theorem 2.1 the following bounds for the number of coincidences (suitably counted) of f_1 and f_2. Assuming that $f_1 \neq \tau^k \circ f_2$ for any integer k, we have:

(i) $\mathrm{ord}(\tau)(\gamma + 1)(d_1 + d_2)$, if X is hyperelliptic.

(ii) $\mathrm{ord}(\tau)\gamma(d_1 + d_2)$, if X is not hyperelliptic.

In particular, if M' is a hyperelliptic surface of genus ≥ 2 and $f_1 \neq J' \circ f_2$, the number of coincidences of f_1 and f_2 is $\leq 2(d_1 + d_2)$ (Theorem 2.5). Moreover, if the image of every point of coincidence is a Weierstrass point of M', the number of coincidences is $\leq d_1 + d_2$ (Corollary 2.9). Both results generalize known facts for automorphisms (see e.g. [F–K] p. 108).

As another consequence of Theorem 2.1, Corollary 2.2 improves slightly a well known result for automorphisms having a sufficiently large number of fixed points ([F–K], p. 265).

Finally in §3, as an application of the results in §2, we obtain a bound for the number of morphisms between two fixed hyperelliptic Riemann surfaces of genera ≥ 2 (Theorem 3.2). The proof is a consequence of the well known result that a morphism between two hyperelliptic surfaces, M and M', maps the Weierstrass points of M into the Weierstrass points of M' ([Mar]), plus the fact, proved in Lemma 3.1, that if two different morphisms coincide on the set of Weierstrass points of M then, of course, they must differ by post-composition with the hyperelliptic involution of M'. This argument is inspired by the classical proof of Schwarz of the finiteness of the group of automorphisms of a compact Riemann surface of genus ≥ 2.

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1. Bounds for the number of coincidences of two different morphisms

Let $P \in M$ be a point and let $f\colon M \to M'$ be a morphism. Usually, one denotes by $m_P(f)$ the multiplicity of P for f, that is, the order of the zero for local parameters around P and $f(P)$. Now, for two morphisms f_1 and f_2 and a coincidence of them, that is, a point $P \in M$ such that $f_1(P) = f_2(P)$, it is possible to extend the usual concept of multiplicity to the multiplicity of a coincidence and, as usual, it does not depend on local parameters:

Definition 1.1: Let $P \in M$ be a coincidence of f_1 and f_2, and let

$$f_1(z) - f_2(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, \quad c_k \neq 0$$

be the Taylor expansion of $f_1 - f_2$ with respect to small parametric discs D of P and D' of $f_i(P)$. We define the **multiplicity** of f_1, f_2 at P to be

$$m_P(f_1, f_2) = k.$$

Notation: In order to simplify the notation, sometimes we will denote the number of coincidences of f_1 and f_2, counted with multiplicities, by $L(f_1, f_2)$, referring to the Lefschetz number for two morphisms with which it agrees (see [F–G]).

Of course, the fixed points of an automorphism T are the coincidences of T and the identity morphism, because for automorphisms the multiplicity of each fixed point is always equal to 1. In this case we will simply write $L(T, Id) = \nu(T)$.

LEMMA 1.2: *Let $f_i\colon M \to M'$ be two different morphisms of degrees d_i $(i = 1, 2)$ between compact Riemann surfaces of genera g and g', respectively; and let $\phi\colon M' \to \mathbf{P}^1$ be a meromorphic function on M'. Assume that $L(f_1, f_2) > \deg(\phi)(d_1 + d_2)$. Then, $\phi \circ f_1 \equiv \phi \circ f_2$ and $d_1 = d_2$.*

Proof: Suppose that $\phi \circ f_1 \neq \phi \circ f_2$. Then, it is clear that

$$\#\{p \in M / f_1(p) = f_2(p)\} \leq \#\{p \in M / \phi \circ f_1(p) = \phi \circ f_2(p)\}.$$

Moreover, $m_p(f_1, f_2) \leq m_p(\phi \circ f_1, \phi \circ f_2) = \operatorname{ord}_p(\phi \circ f_1 - \phi \circ f_2)$, so we have

$$\begin{aligned} \sum_{\{p \in M\colon f_1(p) = f_2(p)\}} m_p(f_1, f_2) &\leq \sum_{\{q \in M\colon \phi \circ f_1(q) = \phi \circ f_2(q)\}} m_q(\phi \circ f_1, \phi \circ f_2) \\ &= \sum_{\{q \in M\colon (\phi \circ f_1 - \phi \circ f_2) q = 0\}} \operatorname{ord}_q(\phi \circ f_1 - \phi \circ f_2) \\ &= \deg(\phi \circ f_1 - \phi \circ f_2) \leq \deg(\phi \circ f_1) + \deg(\phi \circ f_2) \\ &= \deg(\phi)(d_1 + d_2). \end{aligned}$$

Remark: The result for automorphisms appears in [F–K] (p. 261). As we are interested in the number of coincidences of morphisms counted with multiplicities, it was not possible to obtain the result so directly.

Let us now consider the case in which M' admits an automorphism τ, such that the quotient surface is isomorphic to $\mathbf{P}^1$, and let us take as our meromorphic function ϕ the morphism induced by the natural projection, $\phi\colon M' \to M'/ < \tau >$. In this situation we obtain

COROLLARY 1.3: *Let $f_i\colon M \to M'$ be two different morphisms of degrees d_i $(i = 1, 2)$ between compact Riemann surfaces of genera g and g', respectively. Let $\tau\colon M' \to M'$ be an automorphism of M' for which the genus of the quotient suface $M'/ < \tau >$ is zero. Assume $L(f_1, f_2) > \mathrm{ord}(\tau)(d_1+d_2)$. Then $f_1 = \tau^k \circ f_2$, for some integer k, and $L(f_1, f_2) = d_i\nu(\tau^k)$.*

Proof: $L(\tau^k \circ f_2, f_2) = d_i\nu(\tau^k)$ because any point $p \in M$ is a coincidence of f_2 and $\tau^k \circ f_2$, if and only if $f_2(p)$ is a fixed point of τ^k, and $m_p(\tau^k \circ f_2, f_2) = m_p(f_2)$.

The other statements follows directly from Lemma 1.2 taking ϕ as the natural projection to the quotient surface.

Remark 1.4: The above lemma shows that $L(f_1, f_2) \leq \deg(\phi)(d_1 + d_2)$, for any meromorphic function on M', $\phi\colon M' \to \mathbf{P}^1$, such that $\phi \circ f_1 \neq \phi \circ f_2$. Although for a compact Riemann surface M' of genus g', there always exists a meromorphic function of degree $\leq (g'+3)/2$, we cannot use it to obtain a general bound because if we denote such a function by φ, as far as I know, there is no way to ensure that $\varphi \circ f_1 \neq \varphi \circ f_2$. On the other hand, the Riemann–Roch theorem guarantees the existence of a meromorphic function with a unique pole at $p' \in M'$ of degree $\leq g' + 1$, for every $p' \in M'$. Now, for some of these functions, namely ϕ, it is sure that we will have $\phi \circ f_1 \neq \phi \circ f_2$. So, we obtain

$$\sum_{\{p\in M\colon f_1(p)=f_2(p)\}} m_p(f_1, f_2) \leq (g' + 1)(d_1 + d_2)\,, \qquad \text{for any } g'.$$

Now, working with the meromorphic functions associated with the Weierstrass points of M' $(g' \geq 2)$, it is possible to obtain a better result, namely:

THEOREM 1.5: *Let $f_i\colon M \to M'$ be two different morphisms of degrees d_i $(i = 1, 2)$ between compact Riemann surfaces of genera g and $g' \geq 2$. Assume that $L(f_1, f_2) > g'(d_1 + d_2)$, then the following statements must hold:*

(i) $L(f_1, f_2) = (g' + 1)(d_1 + d_2) = (2g' + 2)d_i\,.$

(ii) *M' is a hyperelliptic surface.*

(iii) $f_1 = J' \circ f_2$, *where J' is the hyperelliptic involution of M'.*

Proof: Suppose that

$$L(f_1, f_2) = \sum_{\{p\in M:\ f_1(p)=f_2(p)\}} m_p(f_1, f_2) > g'(d_1 + d_2).$$

Let $W' = \{p'_i\}$ be the set of Weierstrass points of M', and ϕ_i a non-constant meromorphic function of degree $\leq g'$ which has p'_i as its unique pole. From Lemma 1.2, we must have

$$\phi_i \circ f_1 \equiv \phi_i \circ f_2, \qquad \text{for any } i; \tag{1.1}$$

in particular, $d_1 = d_2 = d$.

On the other hand, if we denote by p^i_j $(1 \leq j \leq d_1)$ the points of M such that $f_1(p^i_j) = p'_i$, as p'_i is the only pole of ϕ_i, we obtain

$$\infty = \phi_i(p'_i) = \phi_i \circ f_1(p^i_j) = \phi_i \circ f_2(p^i_j) \Rightarrow f_2(p^i_j) = p'_i, \qquad \text{for any } j.$$

Moreover, (1.1) implies

$$m_{p^i_j}(\phi_i \circ f_1) = \deg(\phi_i) \cdot m_{p^i_j}(f_1) = \deg(\phi_i) \cdot m_{p^i_j}(f_2) = m_{p^i_j}(\phi_i \circ f_2).$$

This means that the points p^i_j are coincidences of f_1 and f_2 with multiplicities

$$m_{p^i_j}(f_1, f_2) \geq m_{p^i_j}(f_1) = m_{p^i_j}(f_2), \qquad \text{for any } i \text{ and } j.$$

Now, applying the fact that the number of Weierstrass points of M', $\#W'$, satisfies $2g' + 2 \leq \#W' \leq g'(g'-1)(g'+1)$ (see e.g. [F–K]), we deduce that

$$\begin{aligned}\sum_{\{p\in M:\ f_1(p)=f_2(p)\}} m_p(f_1, f_2) &\geq \sum_{\{p'_i\in W'\}} \Big(\sum_{\{p^i_j:\ f_1(p^i_j)=p'_i\}} m_{p^i_j}(f_1, f_2)\Big) \\ &\geq \sum_i \Big(\sum_j m_{p^i_j}(f_1)\Big) \\ &\geq (2g'+2)d.\end{aligned}$$

This, together with Remark 1.4, gives

$$\sum_{\{p\in M:\ f_1(p)=f_2(p)\}} m_p(f_1, f_2) = (2g'+2)d,$$

which implies that M' has exactly $2g' + 2$ Weierstrass points or, equivalently, M' is hyperelliptic (see e.g. [F–K]). Now, in the case $\tau = J'$, the hyperelliptic involution of M', Corollary 1.3 tells us that either $f_1 = f_2$ (which is not the case), or $f_1 = J' \circ f_2$, which completes the proof.

Remark 1.6: In [F–G] we obtain the following bound for the number of coincidences, counted with multiplicities, of f_1 and f_2:

$$L(f_1, f_2) \leq d_1 + d_2 + 2g'\sqrt{d_1 d_2}\,.$$

In order to compare it with the bound of Theorem 1.5, one observes that

$$d_1 + d_2 + 2g'\sqrt{d_1 d_2} \leq g'(d_1 + d_2)$$

is equivalent to

$$d_1 + d_2 \leq g'(\sqrt{d_1} - \sqrt{d_2})^2\,.$$

The occurrence of this inequality depends on the relation between g' and the degrees of the morphisms. For instance, when $d_1 = d_2$, that inequality does not happen, which means that the bound of Theorem 1.5 is the smallest one; whereas if $d_1 = 9 \neq d_2 = 4$ and $g' > 13$, the smallest one would be the bound in [F–G].

2. Some particular bounds for surfaces having automorphisms

1 In this section, we rewrite the above results in the particular case in which we have an automorphism τ of M'. If we compose our initial morphisms f_i ($i = 1, 2$) with the morphism which projects to the quotient surface, $\pi\colon M' \to M'/\langle\tau\rangle$, one has

$$L(f_1, f_2) \leq L(\pi \circ f_1, \pi \circ f_2) \leq \operatorname{ord}(\tau)(\gamma + 1)(d_1 + d_2).$$

Combining this with Remark 1.4 and Theorem 1.5, we obtain

THEOREM 2.1: *Let $f_i\colon M \to M'$ be two different morphisms of degrees d_i ($i = 1, 2$) between compact Riemann surfaces of genera g and g', respectively. Let $\tau\colon M' \to M'$ be an automorphism on M', and γ the genus of the quotient surface $X = M'/\langle\tau\rangle$. Then, we have two cases:*

(i) *If X is hyperelliptic and $L(f_1, f_2) > \operatorname{ord}(\tau)(\gamma+1)(d_1+d_2)$, then $f_1 = \tau^k \circ f_2$ and $L(f_1, f_2) = d_i\nu(\tau^k)$.*

(ii) *If X is not hyperelliptic and $L(f_1, f_2) > \operatorname{ord}(\tau)\gamma(d_1+d_2)$, then $f_1 = \tau^k \circ f_2$ and $L(f_1, f_2) = d_i\nu(\tau^k)$.*

A standard technique to obtain information about the number of fixed points of an automorphism τ of a surface M is to apply the Riemann–Hurwitz formula to the natural projection $M \to M/\langle\tau\rangle$. Our Theorem 2.1 permits us to obtain, in fact to improve, one of these well known results (see [F–K], p. 265):

COROLLARY 2.2: *Let M be a compact surface of genus g. Let $1 \neq \tilde{T} \in \mathrm{Aut}(M)$ with $n = \mathrm{ord}(\tilde{T})$, and let $\tilde{g}$ be the genus of the quotient surface $\tilde{M} = M/\langle\tilde{T}\rangle$. Suppose there exists another $1 \neq T \in \mathrm{Aut}(M)$ with $\nu(T) > 2n(\tilde{g}+1)$. Then, the following statements hold:*

(i) *$T \in \langle\tilde{T}\rangle$; in particular, each fixed point of $\tilde{T}$ is a fixed point of T.*
(ii) *If n is a prime, $\langle\tilde{T}\rangle$ is a normal subgroup of $\mathrm{Aut}(M)$.*
(iii) *If $n = 2$, then $\langle\tilde{T}\rangle$ is in the center of $\mathrm{Aut}(M)$.*

Proof: The first statement follows directly from Theorem 2.1, because $\nu(T) = L(T, \mathrm{Id}) \leq L(\pi\circ T, \pi)$ if $\pi\circ T \neq \pi$, where $\pi\colon M \to M/\langle\tilde{T}\rangle$ is the natural projection.

On the other hand, if the order of $\tilde{T}$ is a prime, applying the Riemann–Hurwitz formula to the morphism π, one obtains that $\nu(\tilde{T}^k) = \nu(\tilde{T})$, for any k, because $\langle\tilde{T}^k\rangle = \langle\tilde{T}\rangle$. In particular, $\nu(\tilde{T}) = \nu(T) > 2n(\tilde{g}+1)$. As for any $T \in \mathrm{Aut}(M)$ one has $L(T\circ\tilde{T}^k, T) = \nu(\tilde{T}^k)$, Theorem 2.1 gives us $T\circ\tilde{T}^k = \tilde{T}^j \circ T$, which finishes the proof.

Remark 2.3: Observe that for n prime, (i) and the assumption $\nu(T) > 2n(\tilde{g}+1)$ give us, from the Riemann–Hurwitz formula, that $g > n^2\tilde{g} + (n-1)^2$ because $\nu(T) = \nu(\tilde{T})$, which is a hypothesis in [F-K] Theorem V.1.8. Furthermore, we prove not only that the fixed point set of $\tilde{T}$ is contained in the fixed point set of T, but that, in fact, T is a power of $\tilde{T}$, for $n = \mathrm{ord}(\tilde{T})$ not necessarily a prime number.

The classical consequences of this result are:

(1) Any automorphism on a hyperelliptic surface M of genus $g \geq 2$ which fixes more than 4 ($= 2\cdot 2(0+1)$) points must be the hyperelliptic involution, which is well known to commute with every element of $\mathrm{Aut}(M)$.

In our context, the necessity of assuming $g \geq 2$ comes from the fact that the existence of $T \in \mathrm{Aut}(M)$ with $4 < \nu(T) \leq 2g+2$ implies $g \geq 2$.

(2) Any automorphism on a γ-hyperelliptic surface M of genus $g > 4\gamma+1$ which fixes more than $4(\gamma+1)$ must be the γ-hyperelliptic involution, which we denote by J_γ. Moreover, J_γ is in the center of $\mathrm{Aut}(M)$ (see e.g. [F–K], p. 266).

In this case, the condition $g > 4\gamma+1$ comes from a similar fact, namely, the existence of $T \in \mathrm{Aut}(M)$ with $2\cdot 2(\gamma+1) < \nu(T) \leq 2g+2$ implies that $g \geq 4\gamma+1$.

COROLLARY 2.4: *Let $f\colon M \to M'$ be a morphism of degree d between compact Riemann surfaces of genera g and g', respectively. Suppose there exist $1 \neq T \in \mathrm{Aut}(M)$ and $1 \neq T' \in \mathrm{Aut}(M')$ with the quotient surface, $X' = M'/\langle T'\rangle$, of genus γ, and that $\nu(T) > 2d\cdot \mathrm{ord}\,T'(\gamma+1)$. Then, $f\circ T = {T'}^k \circ f$ for some integer k.*

Proof: Obviously, $\nu(T) \leq L(f \circ T, f)$, so Theorem 2.1 gives us directly the result.

2. If M' is a hyperelliptic surface, that is, M' admits an automorphism J' of order 2 with respect to which the quotient surface has genus 0, then Theorem 2.1 gives

THEOREM 2.5: *Let f_i: $M \to M'$ be two different morphisms of degrees d_i ($i = 1, 2$) between compact Riemann surfaces of genera g and $g' \geq 2$, and let us assume that M' is hyperelliptic with hyperelliptic involution J'. Then, either*

(1) *$L(f_1, f_2) \leq 2(d_1 + d_2)$; or*
(2) *$L(f_1, f_2) = (2 + 2g')d_i$ and $f_1 = J' \circ f_2$.*

Remark 2.6: The above theorem generalizes the well known result that any automorphism of a hyperelliptic surface fixes at most 4 points, except for the hyperelliptic involution which fixes $2g + 2$ points (see e.g. [F–K], p. 108).

Now, we describe an example for which bound (1) of Theorem 2.5 is attained. An example reading the bound (2) is presented in [F–G].

Example 2.7: Let M and M' be the hyperelliptic curves given by the following algebraic equations: $y^2 = x^{2n} - 1$ and $y^2 = x^n - 1$, with n even, and let f_j: $M \to M'$ ($0 \leq j \leq n - 1$) be the morphisms defined by

$$f_j(x, y) = (\xi^j x^2, y); \quad \text{where } \xi = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

One has genus$(M) = n - 1$, genus$(M') = (n - 2)/2$ and $\deg(f_k) = 2$, for any k.

Obviously, the points on M $(0, \pm\sqrt{-1})$ are coincidences of f_0 and f_k, for any $k \neq 0$. In order to compute their multiplicities, we look at the local expression of f_0 around any of those points:

$$s \to (s, \sqrt{s^{2n} - 1}) \to (s^2, \sqrt{s^{2n} - 1}) \to s^2.$$

Similarly, the local expression of $f_k (k \neq 0)$ is

$$s \longrightarrow (s, \sqrt{s^{2n} - 1}) \longrightarrow (\xi^k s^2, \sqrt{s^{2n} - 1}) \longrightarrow \xi^k s^2.$$

Thus we see that, locally, one has $(f_0 - f_k)(s) = (1 - \xi^k)s^2$; so the multiplicity of those points, as coincidences of f_0 and f_k, is 2.

We denote by ∞_1 and ∞_2 the two points at infinity of M, and by ∞_1' and ∞_2' those of M'.

The local expression of f_0 around $\infty_1 \in M$ is

$$s \longrightarrow \left(\frac{1}{s}, \frac{\sqrt{1-s^{2n}}}{s^n}\right) \longrightarrow \left(\frac{1}{s^2}, \frac{\sqrt{1-(s^2)^n}}{(s^2)^{n/2}}\right) \longrightarrow s^2,$$

and the local expression of f_k around $\infty_1 \in M$ is

$$s \longrightarrow \left(\frac{1}{s}, \frac{\sqrt{1-s^{2n}}}{s^n}\right) \longrightarrow \left(\frac{\xi^k}{s^2}, \frac{\sqrt{1-(s^2)^n}}{(s^2)^{n/2}}\right) \longrightarrow \frac{s^2}{\xi^k}.$$

One has $\xi^{n/2} = -1$ because we are assuming that n is even, so

$$(\xi^k)^{n/2} = \begin{cases} 1, & \text{for } k \text{ even,} \\ -1, & \text{for } k \text{ odd.} \end{cases}$$

That implies, for k an even number,

$$\left(\frac{\xi^k}{s^2}, \frac{\sqrt{1-(s^2)^n}}{(s^2)^{n/2}}\right) = \left(\frac{1}{(s^2/\xi^k)}, \frac{\sqrt{1-(s^2/\xi^k)^n}}{(s^2/\xi^k)^{n/2}}\right)$$

which is a neighbourhood of ∞_1; and for k an odd number,

$$\left(\frac{\xi^k}{s^2}, \frac{\sqrt{1-(s^2)^n}}{(s^2)^{n/2}}\right) = \left(\frac{1}{(s^2/\xi^k)}, -\frac{\sqrt{1-(s^2/\xi^k)^n}}{(s^2/\xi^k)^{n/2}}\right)$$

which is a neighbourhood of ∞_2.

Summing up, for k a non-zero even number, ∞_1 is a coincidence of f_0 and f_k with multiplicity equal to 2. Similarly for $\infty_2 \in M$. So, one has

$$\sum_{\{p:\, f_0(p)=f_k(p)\}} m_p(f_0, f_k) = 8 = 2(d_0 + d_k).$$

Working in a similar way as in the proof of Theorem 1.5, it is possible to generalize another result about fixed points of automorphisms of hyperelliptic surfaces of genera bigger than or equal to 2, which claims that any such automorphism, different from the hyperelliptic involution, can only fix 2 points if both of them are Weierstrass points, or 3 points in the case that just one of the fixed points is a Weierstrass point (see e.g. [F–K], p. 109):

LEMMA 2.8: *Let M' be a hyperelliptic surface of genus $g' \geq 2$, and $f_i\colon M \to M'$ $(i = 1, 2)$ two different morphisms of degrees d_i such that $f_1 \neq J' \circ f_2$, where J' is the hyperelliptic involution of M'. Let $q \in M$ be a coincidence of f_1 and f_2 such that $f_i(q) = q'$ is a Weierstrass point of M'. Then,*

$$2 \cdot m_q(f_1, f_2) + \sum_{\{q \neq p \in M\colon f_1(p) = f_2(p)\}} m_p(f_1, f_2) \leq 2(d_1 + d_2).$$

Proof: Let $x\colon M' \to \mathbf{P}^1$ be the hyperelliptic function. It is well known that x ramifies on every Weierstrass point of M'. In particular $m_{q'}(x) = 2$, and $\mathrm{ord}_q(x \circ f_1 - x \circ f_2) = m_q(x \circ f_1, x \circ f_2) = m_{q'}(x) \cdot m_q(f_1, f_2)$. For any other coincidence $p \neq q$, we have $m_p(f_1, f_2) \leq \mathrm{ord}_p(x \circ f_1 - x \circ f_2)$, which give us

$$\begin{aligned} 2m_q(f_1, f_2) + \sum_{\{q \neq p \in M\colon f_1(p) = f_2(p)\}} m_p(f_1, f_2) &\leq \mathrm{ord}_q(x \circ f_1 - x \circ f_2) \\ &+ \sum_{\{q \neq p \in M\colon x \circ f_1(p) = x \circ f_2(p)\}} \mathrm{ord}_p(x \circ f_1 - x \circ f_2) \\ &= \deg(x \circ f_1 - x \circ f_2) \leq 2(d_1 + d_2), \end{aligned}$$

as we wanted.

COROLLARY 2.9: *Let M' be a hyperelliptic surface of genus $g' \geq 2$, and $f_i\colon M \to M'$ $(i = 1, 2)$ two different morphisms of degrees d_i, such that $f_1 \neq J' \circ f_2$. Assume that the image of every coincidence of f_1 and f_2 is a Weierstrass point of M'. Then*

$$L(f_1, f_2) \leq d_1 + d_2.$$

The following result is a corollary of Remark 1.4, Theorem 1.5 and Theorem 2.5:

COROLLARY 2.10: *Let $f\colon M \to M'$ be a morphism of degree d between compact Riemann surfaces of genera g and g', and let T be an automorphism of M.*

(i) *If $g' \leq 1$ and $\nu(T) > 2d(g' + 1)$, then $f \circ T = f$.*
(ii) *If $g' > 2$, M' is not hyperelliptic and $\nu(T) > 2d \cdot g'$, then $f \circ T = f$.*
(iii) *If $g' \geq 2$, M' is hyperelliptic and $\nu(T) > 4d$, then we must have either $f \circ T = f$, or $f \circ T = J' \circ f$ where J' is the hyperelliptic involution of M'.*

Proof: Obviously $L(f \circ T, f) \geq \nu(T)$, so statement (i) follows from Remark 1.4; for statement (ii) one applies Theorem 1.5; and, finally, Theorem 2.5 gives statement (iii).

Remark 2.11: In other words, the above statements claim that in such conditions, the morphism f induces a morphism between the quotient surface $M/\langle T\rangle$ and M'. In particular, statement (i) can be seen as a generalization of the following result (see e.g. [Ac], p. 20): "Let $f\colon M \to \mathbf{P}^1$ be a meromorphic function of degree $\leq g$ from a hyperelliptic surface of genus g. Let $\pi\colon M \to \mathbf{P}^1$ be a 2-sheeted cover. Then there exists a rational function R satisfying $f = R \circ \pi$."

3. Morphisms between hyperelliptic surfaces

Schwarz gave an elegant and simple argument of finiteness of the number of automorphisms of compact Riemann surface of genus bigger than or equal to 2, based on two facts:

(1) An automorphism sends Weierstrass points to Weierstrass points.

(2) If two automorphisms agree on every Weierstrass point, they must be equal or differ by composition with the hyperelliptic involution.

In the case of morphisms, it is well known that if one has a morphism from a hyperelliptic surface M, $f\colon M \to M'$, then the image surface M' must be hyperelliptic too (see [Mar]). Moreover, f must send the $2g+2$ Weierstrass points of M among the $2g'+2$ Weierstrass points of M'.

We now give the analogue of (2) for morphisms between fixed hyperelliptic surfaces. This will permit us to obtain a bound for the number of such morphisms, thereby providing in this particular case a different proof of the classical theorem of de Franchis.

LEMMA 3.1: *Let $f_i\colon M \to M'$ $(i = 1,2)$ be two morphisms of degrees d_i between compact hyperelliptic Riemann surfaces of genera g and $g' \geq 2$, such that they have every Weierstrass point of M as a coincidence. Then, one has $f_1 = f_2$, or $f_1 = J' \circ f_2$, where J' is the hyperelliptic involution of M'.*

Proof: Every hyperelliptic Riemann surface of genus $g \geq 2$ has $2g+2$ Weierstrass points, so if we assume that $f_1 \neq f_2$, and they agree on every Weierstrass point of M, Lemma 2.8 implies that we have either

$$2(2g+2) \leq 2(d_1 + d_2),$$

or $f_1 = J' \circ f_2$. Now, the Riemann–Hurwitz formula gives us

$$d_i \leq \frac{g-1}{g'-1} \leq g - 1 < g + 1,$$

so $d_1 + d_2 < 2g + 2$, which implies that $f_1 = J' \circ f_2$. This completes the proof.

THEOREM 3.2: *The number of morphisms between two fixed hyperelliptic surfaces of genus g and $g' \geq 2$ is bounded by*

$$2(2g'+2)^{2g+2}.$$

Remark: The above argument is not valid for morphisms from a non-hyperelliptic surface M, because it is not true, in general, that a morphism maps Weierstrass points to Weierstrass points, as we can see in the following example:

Example 3.3: Let M be the Fermat curve given by the algebraic equation $x^{2n} + y^{2n} = 1$, and M' the hyperelliptic curve $y^2 = 1 - x^{2n}$. One has the morphism

$$f\colon M \to M'$$
$$(x, y) \to (x, y^n).$$

We denote by σ the following automorphism of M:

$$\sigma(x, y) = (\xi x, y), \qquad \text{where } \xi = \exp\left(\frac{\pi\sqrt{-1}}{n}\right).$$

$\{(0, \xi^k)\}_{k=1}^{2n}$ are obviously fixed points of σ, and for $n > 2$ they must be Weierstrass points of M, from a well known result of Lewittes which claims that every fixed point of an automorphism of a compact Riemann surface of genus bigger than or equal to 2 which fixes more than 4 points is a Weierstrass point ([Le] or [F–K] p. 264).

On the other hand, $f(0, \xi^k) = (0, \xi^{nk})$, which is not a Weierstrass point for the Weierstrass points of M' are $\{(\xi^k, 0)\}_{k=1}^{2n}$.

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